

Bipolar-Valued Fuzzy Ideals in LA-semigroups

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Abstract

In this paper, the notion of bipolar-valued fuzzy LA-subsemigroups is introduced and also some properties of bipolar-valued fuzzy left (right, bi-, interior) ideals of LA-semigroups has been discussed.

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1 Introduction

The concept of a fuzzy set was introduced by Zadeh [13], in 1965. Since its inception, the theory has developed in many directions and found applications in a wide variety of fields. There has been a rapid growth in the interest of fuzzy set theory and its applications from the past several years. Many researchers published high-quality research articles on fuzzy sets in a variety of international journals. The study of fuzzy set in algebraic structure has been started in the definitive paper of Rosenfeld 1971 [11]. Fuzzy subgroup and its important properties were defined and established by Rosenfeld [11]. In 1981, Kuroki introduced the concept of fuzzy ideals and fuzzy bi-ideals in semigroups in his paper [4].

There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, etc. Bipolar-valued fuzzy set is another extension of fuzzy set whose membership degree range is different from the above extensions. Lee [5] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. In a bipolar-valued fuzzy set, the membership degree 0 indicate that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ assign that elements somewhat satisfy the property, and the membership degrees on $[-1, 0)$ assign that elements somewhat satisfy the implicit counter-property [5, 6].

Akram et al. [1] introduced the concept of bipolar fuzzy K-algebras. In [2], Jun and park applied the notion of bipolar-valued fuzzy sets to BCH-algebras. They introduced the concept of bipolar fuzzy subalgebras and bipolar fuzzy ideals of a BCH-algebra. Lee [7] applied the notion of bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras. Also some results on bipolar-valued fuzzy BCK/BCI-algebras are introduced by Saeid in [12].

This paper concerns the relationship between bipolar-valued fuzzy sets and left almost semigroups. The left almost semigroup abbreviated as an LA-semigroup, was first introduced by Kazim and Naseerudin [3]. They generalized some useful results of semigroup theory. They introduced braces on the left of the ternary commutative law $abc = cba$, to get a new pseudo associative law, that is $(ab)c = (cb)a$, and named it as left invertive law. An LA-semigroup is the midway structure between a commutative semigroup and a groupoid. Despite the fact, the structure is non-associative and non-commutative. It nevertheless possesses many interesting properties which we usually find in commutative and associative algebraic structures. Mushtaq and Yusuf produced useful results [9], on locally associative LA-semigroups in 1979. In this structure they defined powers of an element and congruences using these powers. They constructed quotient LA-semigroups using these congruences. It is a useful non-associative structure with wide applications in theory of flocks.

In this paper, we have introduced the notion of bipolar-valued fuzzy LA-subsemigroups and bipolar-valued fuzzy left (right, bi-, interior) ideals in LA-semigroups.

2 Preliminaries and basic definitions

Definition 2.1. [3] A groupoid (S, \cdot) is called an LA-semigroup, if it satisfies left invertive law

$$(a \cdot b) \cdot c = (c \cdot b) \cdot a, \text{ for all } a, b, c \in S.$$

Example 2.1 [8] Let $(\mathbb{Z}, +)$ denote the commutative group of integers under addition. Define a binary operation “ $*$ ” in \mathbb{Z} as follows:

$$a * b = b - a, \text{ for all } a, b \in \mathbb{Z}.$$

Where “ $-$ ” denotes the ordinary subtraction of integers. Then $(\mathbb{Z}, *)$ is an LA-semigroup.

Example 2.2 [8] Define a binary operation “ $*$ ” in \mathbb{R} as follows:

$$a * b = b \div a, \text{ for all } a, b \in \mathbb{R}.$$

Then $(\mathbb{R}, *)$ is an LA-semigroup.

Lemma 2.1 [9] *If S is an LA-semigroup with left identity e , then $a(bc) = b(ac)$ for all $a, b, c \in S$.*

Let S be an LA-semigroup. A nonempty subset A of S is called an LA-subsemigroup of S if $ab \in A$ for all $a, b \in A$. A nonempty subset L of S is called a left ideal of S if $SL \subseteq L$ and a nonempty subset R of S is called a right ideal of S if $RS \subseteq R$. A nonempty subset I of S is called an ideal of S if I is both a left and a right ideal of S . A subset A of S is called an interior ideal of S if $(SA)S \subseteq A$. An LA-subsemigroup A of S is called a bi-ideal of S if $(AS)A \subseteq A$.

In an LA-semigroup the medial law holds:

$$(ab)(cd) = (ac)(bd), \text{ for all } a, b, c, d \in S.$$

In an LA-semigroup S with left identity, the paramedial law holds:

$$(ab)(cd) = (dc)(ba), \quad \text{for all } a, b, c, d \in S.$$

Now we will recall the concept of bipolar-valued fuzzy sets.

Definition 2.2 [6] Let X be a nonempty set. A bipolar-valued fuzzy subset (BVF-subset, in short) B of X is an object having the form

$$B = \{ \langle x, \mu_B^+(x), \mu_B^-(x) \rangle : x \in X \}.$$

Where $\mu_B^+ : X \rightarrow [0, 1]$ and $\mu_B^- : X \rightarrow [-1, 0]$.

The positive membership degree $\mu_B^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $B = \{ \langle x, \mu_B^+(x), \mu_B^-(x) \rangle : x \in X \}$, and the negative membership degree $\mu_B^-(x)$ denotes the satisfaction degree of x to some implicit counter property of $B = \{ \langle x, \mu_B^+(x), \mu_B^-(x) \rangle : x \in X \}$. For the sake of simplicity, we shall use the symbol $B = \langle \mu_B^+, \mu_B^- \rangle$ for the bipolar-valued fuzzy set $B = \{ \langle x, \mu_B^+(x), \mu_B^-(x) \rangle : x \in X \}$.

Definition 2.3 Let $B_1 = \langle \mu_{B_1}^+, \mu_{B_1}^- \rangle$ and $B_2 = \langle \mu_{B_2}^+, \mu_{B_2}^- \rangle$ be two BVF-subsets of a nonempty set X . Then the product of two BVF-subsets is denoted by $B_1 \circ B_2$ and defined as:

$$\begin{aligned} (\mu_{B_1}^+ \circ \mu_{B_2}^+)(x) &= \begin{cases} \bigvee_{x=yz} \{ \mu_{B_1}^+(y) \wedge \mu_{B_2}^+(z) \}, & \text{if } x = yz \text{ for some } y, z \in S \\ 0 & \text{otherwise.} \end{cases} \\ (\mu_{B_1}^- \circ \mu_{B_2}^-)(x) &= \begin{cases} \bigwedge_{x=yz} \{ \mu_{B_1}^-(y) \vee \mu_{B_2}^-(z) \}, & \text{if } x = yz \text{ for some } y, z \in S \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that an LA-semigroup S can be considered as a BVF-subset of itself and let

$$\begin{aligned} \Gamma &= \langle \mathcal{S}_\Gamma^+(x), \mathcal{S}_\Gamma^-(x) \rangle \\ &= \{ \langle x, \mathcal{S}_\Gamma^+(x), \mathcal{S}_\Gamma^-(x) \rangle : \mathcal{S}_\Gamma^+(x) = 1 \text{ and } \mathcal{S}_\Gamma^-(x) = -1, \text{ for all } x \text{ in } S \} \end{aligned}$$

be a BVF-subset and $\Gamma = \langle \mathcal{S}_\Gamma^+(x), \mathcal{S}_\Gamma^-(x) \rangle$ will be carried out in operations with a BVF-subset $B = \langle \mu_B^+, \mu_B^- \rangle$ such that \mathcal{S}_Γ^+ and \mathcal{S}_Γ^- will be used in collaboration with μ_B^+ and μ_B^- respectively.

Let $BVF(S)$ denote the set of all BVF-subsets of an LA-semigroup S .

Proposition 2.2 Let S be an LA-semigroup, then the set $(BVF(S), \circ)$ is an LA-semigroup.

Proof. Clearly $BVF(S)$ is closed. Let $B_1 = \langle \mu_{B_1}^+, \mu_{B_1}^- \rangle$, $B_2 = \langle \mu_{B_2}^+, \mu_{B_2}^- \rangle$ and $B_3 = \langle \mu_{B_3}^+, \mu_{B_3}^- \rangle$ be in $BVF(S)$. Let x be any element of S such that $x \neq yz$ for some $y, z \in S$. Then we have

$$((\mu_{B_1}^+ \circ \mu_{B_2}^+) \circ \mu_{B_3}^+)(x) = 0 = ((\mu_{B_3}^+ \circ \mu_{B_2}^+) \circ \mu_{B_1}^+)(x).$$

And

$$((\mu_{B_1}^- \circ \mu_{B_2}^-) \circ \mu_{B_3}^-)(x) = 0 = ((\mu_{B_3}^- \circ \mu_{B_2}^-) \circ \mu_{B_1}^-)(x).$$

Let x be any element of S such that $x = yz$ for some $y, z \in S$. Then we have

$$\begin{aligned}
((\mu_{B_1}^+ \circ \mu_{B_2}^+) \circ \mu_{B_3}^+)(x) &= \bigvee_{x=yz} \{(\mu_{B_1}^+ \circ \mu_{B_2}^+)(y) \wedge \mu_{B_3}^+(z)\} \\
&= \bigvee_{x=yz} \left\{ \left(\bigvee_{y=pq} \{\mu_{B_1}^+(p) \wedge \mu_{B_2}^+(q)\} \right) \wedge \mu_{B_3}^+(z) \right\} \\
&= \bigvee_{x=yz} \bigvee_{y=pq} \{\mu_{B_1}^+(p) \wedge \mu_{B_2}^+(q) \wedge \mu_{B_3}^+(z)\} \\
&= \bigvee_{x=(pq)z} \{\mu_{B_1}^+(p) \wedge \mu_{B_2}^+(q) \wedge \mu_{B_3}^+(z)\} \\
&= \bigvee_{x=(zq)p} \{\mu_{B_3}^+(z) \wedge \mu_{B_2}^+(q) \wedge \mu_{B_1}^+(p)\} \\
&= \bigvee_{x=sp} \left\{ \left(\bigvee_{s=zq} \{\mu_{B_3}^+(z) \wedge \mu_{B_2}^+(q)\} \right) \wedge \mu_{B_1}^+(p) \right\} \\
&= \bigvee_{x=sp} \{(\mu_{B_3}^+ \circ \mu_{B_2}^+)(s) \wedge \mu_{B_1}^+(p)\} \\
&= ((\mu_{B_3}^+ \circ \mu_{B_2}^+) \circ \mu_{B_1}^+)(x).
\end{aligned}$$

And

$$\begin{aligned}
((\mu_{B_1}^- \circ \mu_{B_2}^-) \circ \mu_{B_3}^-)(x) &= \bigwedge_{x=yz} \{(\mu_{B_1}^- \circ \mu_{B_2}^-)(y) \vee \mu_{B_3}^-(z)\} \\
&= \bigwedge_{x=yz} \left\{ \left(\bigwedge_{y=pq} \{\mu_{B_1}^-(p) \vee \mu_{B_2}^-(q)\} \right) \vee \mu_{B_3}^-(z) \right\} \\
&= \bigwedge_{x=yz} \bigwedge_{y=pq} \{\mu_{B_1}^-(p) \vee \mu_{B_2}^-(q) \vee \mu_{B_3}^-(z)\} \\
&= \bigwedge_{x=(pq)z} \{\mu_{B_1}^-(p) \vee \mu_{B_2}^-(q) \vee \mu_{B_3}^-(z)\} \\
&= \bigwedge_{x=(zq)p} \{\mu_{B_3}^-(z) \vee \mu_{B_2}^-(q) \vee \mu_{B_1}^-(p)\} \\
&= \bigwedge_{x=sp} \left\{ \left(\bigwedge_{s=zq} \{\mu_{B_3}^-(z) \vee \mu_{B_2}^-(q)\} \right) \vee \mu_{B_1}^-(p) \right\} \\
&= \bigwedge_{x=sp} \{(\mu_{B_3}^- \circ \mu_{B_2}^-)(s) \vee \mu_{B_1}^-(p)\} \\
&= ((\mu_{B_3}^- \circ \mu_{B_2}^-) \circ \mu_{B_1}^-)(x).
\end{aligned}$$

Hence $(BVF(S), \circ)$ is an LA-semigroup. \square

Corollary 2.3 *If S is an LA-semigroup, then the medial law holds in $BVF(S)$.*

Proof. Let $B_1 = \langle \mu_{B_1}^+, \mu_{B_1}^- \rangle$, $B_2 = \langle \mu_{B_2}^+, \mu_{B_2}^- \rangle$, $B_3 = \langle \mu_{B_3}^+, \mu_{B_3}^- \rangle$ and $B_4 = \langle \mu_{B_4}^+, \mu_{B_4}^- \rangle$ be in $BVF(S)$. By successive use of left invertive law

$$\begin{aligned}
(\mu_{B_1}^+ \circ \mu_{B_2}^+) \circ (\mu_{B_3}^+ \circ \mu_{B_4}^+) &= ((\mu_{B_3}^+ \circ \mu_{B_4}^+) \circ \mu_{B_2}^+) \circ \mu_{B_1}^+ \\
&= ((\mu_{B_2}^+ \circ \mu_{B_4}^+) \circ \mu_{B_3}^+) \circ \mu_{B_1}^+ \\
&= (\mu_{B_1}^+ \circ \mu_{B_3}^+) \circ (\mu_{B_2}^+ \circ \mu_{B_4}^+).
\end{aligned}$$

And

$$\begin{aligned}
(\mu_{B_1}^- \circ \mu_{B_2}^-) \circ (\mu_{B_3}^- \circ \mu_{B_4}^-) &= ((\mu_{B_3}^- \circ \mu_{B_4}^-) \circ \mu_{B_2}^-) \circ \mu_{B_1}^- \\
&= ((\mu_{B_2}^- \circ \mu_{B_4}^-) \circ \mu_{B_3}^-) \circ \mu_{B_1}^- \\
&= (\mu_{B_1}^- \circ \mu_{B_3}^-) \circ (\mu_{B_2}^- \circ \mu_{B_4}^-).
\end{aligned}$$

Hence this shows that the medial law holds in $BVF(S)$. \square

3 Bipolar-valued fuzzy ideals in LA-semigroup

Definition 3.1 A BVF-subset $B = \langle \mu_B^+, \mu_B^- \rangle$ of an LA-semigroup S is called a bipolar-valued fuzzy LA-subsemigroup of S if

$$\mu_B^+(xy) \geq \mu_B^+(x) \wedge \mu_B^+(y) \quad \text{and} \quad \mu_B^-(xy) \leq \mu_B^-(x) \vee \mu_B^-(y)$$

for all $x, y \in S$.

Definition 3.2 A BVF-subset $B = \langle \mu_B^+, \mu_B^- \rangle$ of an LA-semigroup S is called a bipolar-valued fuzzy left ideal of S if

$$\mu_B^+(xy) \geq \mu_B^+(y) \quad \text{and} \quad \mu_B^-(xy) \leq \mu_B^-(y)$$

for all $x, y \in S$.

Definition 3.3 A BVF-subset $B = \langle \mu_B^+, \mu_B^- \rangle$ of an LA-semigroup S is called a bipolar-valued fuzzy right ideal of S if

$$\mu_B^+(xy) \geq \mu_B^+(x) \quad \text{and} \quad \mu_B^-(xy) \leq \mu_B^-(x)$$

for all $x, y \in S$.

A BVF-subset $B = \langle \mu_B^+, \mu_B^- \rangle$ of an LA-semigroup S is called a BVF-ideal or BVF-two-sided ideal of S if $B = \langle \mu_B^+, \mu_B^- \rangle$ is both BVF-left and BVF-right ideal of S .

Example 3.1 Let $S = \{a, b, c, d\}$, the binary operation "·" on S be defined as follows:

| · | a | b | c | d |
|---|---|---|---|---|
| a | b | d | c | a |
| b | a | b | c | d |
| c | c | c | c | c |
| d | d | a | c | b |

Clearly, S is an LA-semigroup. But S is not a semigroup because $d = d \cdot (b \cdot a) \neq (d \cdot b) \cdot a = b$. Now we define BVF-subset as

$$B = \langle \mu_B^+, \mu_B^- \rangle = \left\langle \left(\frac{a}{0.2}, \frac{b}{0.2}, \frac{c}{0.7}, \frac{d}{0.2} \right), \left(\frac{a}{-0.5}, \frac{b}{-0.5}, \frac{c}{-0.8}, \frac{d}{-0.5} \right) \right\rangle.$$

Clearly B is a BVF-ideal of S .

Proposition 3.1 Every BVF-left (BVF-right) ideal $B = \langle \mu_B^+, \mu_B^- \rangle$ of an LA-semigroup S is a bipolar-valued fuzzy LA-subsemigroup of S .

Proof. Let $B = \langle \mu_B^+, \mu_B^- \rangle$ be a BVF-left ideal of S and for any $x, y \in S$,

$$\mu_B^+(xy) \geq \mu_B^+(y) \geq \mu_B^+(x) \wedge \mu_B^+(y).$$

And

$$\mu_B^-(xy) \leq \mu_B^-(y) \leq \mu_B^-(x) \vee \mu_B^-(y).$$

Hence $B = \langle \mu_B^+, \mu_B^- \rangle$ is a bipolar-valued fuzzy LA-subsemigroup of S . The other case can be prove in a similar way. \square

Lemma 3.2 *Let $B = \langle \mu_B^+, \mu_B^- \rangle$ be a BVF-subset of an LA-semigroup S . Then*

(1) $B = \langle \mu_B^+, \mu_B^- \rangle$ is a BVF-LA-subsemigroup of S if and only if $\mu_B^+ \circ \mu_B^+ \subseteq \mu_B^+$ and $\mu_B^- \circ \mu_B^- \supseteq \mu_B^-$.

(2) $B = \langle \mu_B^+, \mu_B^- \rangle$ is a BVF-left (resp. BVF-right) ideal of S if and only if $\mathcal{S}_\Gamma^+ \circ \mu_B^+ \subseteq \mu_B^+$ and $\mathcal{S}_\Gamma^- \circ \mu_B^- \supseteq \mu_B^-$ (resp. $\mu_B^+ \circ \mathcal{S}_\Gamma^+ \subseteq \mu_B^+$ and $\mu_B^- \circ \mathcal{S}_\Gamma^- \supseteq \mu_B^-$).

Proof. (1) Let $B = \langle \mu_B^+, \mu_B^- \rangle$ be a BVF-LA-subsemigroup of S and $x \in S$. If $(\mu_B^+ \circ \mu_B^+)(x) = 0$ and $(\mu_B^- \circ \mu_B^-)(x) = 0$, then $(\mu_B^+ \circ \mu_B^+)(x) \leq \mu_B^+(x)$ and $(\mu_B^- \circ \mu_B^-)(x) \geq \mu_B^-(x)$. Otherwise,

$$(\mu_B^+ \circ \mu_B^+)(x) = \bigvee_{x=yz} \{\mu_B^+(y) \wedge \mu_B^+(z)\} \leq \bigvee_{x=yz} \mu_B^+(yz) = \mu_B^+(x).$$

And

$$(\mu_B^- \circ \mu_B^-)(x) = \bigwedge_{x=yz} \{\mu_B^-(y) \vee \mu_B^-(z)\} \geq \bigwedge_{x=yz} \mu_B^-(yz) = \mu_B^-(x).$$

Thus $\mu_B^+ \circ \mu_B^+ \subseteq \mu_B^+$ and $\mu_B^- \circ \mu_B^- \supseteq \mu_B^-$.

Conversely, let $\mu_B^+ \circ \mu_B^+ \subseteq \mu_B^+$, $\mu_B^- \circ \mu_B^- \supseteq \mu_B^-$ and $x, y \in S$, then

$$\mu_B^+(xy) \geq (\mu_B^+ \circ \mu_B^+)(xy) = \bigvee_{xy=ab} \{\mu_B^+(a) \wedge \mu_B^+(b)\} \geq \mu_B^+(x) \wedge \mu_B^+(y).$$

And

$$\mu_B^-(xy) \leq (\mu_B^- \circ \mu_B^-)(xy) = \bigwedge_{xy=ab} \{\mu_B^-(a) \vee \mu_B^-(b)\} \leq \mu_B^-(x) \vee \mu_B^-(y).$$

So $B = \langle \mu_B^+, \mu_B^- \rangle$ is a BVF-LA-subsemigroup of S .

(2) Let $B = \langle \mu_B^+, \mu_B^- \rangle$ be a BVF-left ideal of S and $x \in S$. If $(\mathcal{S}_\Gamma^+ \circ \mu_B^+)(x) = 0$ and $(\mathcal{S}_\Gamma^- \circ \mu_B^-)(x) = 0$, then $(\mathcal{S}_\Gamma^+ \circ \mu_B^+)(x) \leq \mu_B^+(x)$ and $(\mathcal{S}_\Gamma^- \circ \mu_B^-)(x) \geq \mu_B^-(x)$. Otherwise,

$$\begin{aligned} (\mathcal{S}_\Gamma^+ \circ \mu_B^+)(x) &= \bigvee_{x=ab} \{\mathcal{S}_\Gamma^+(a) \wedge \mu_B^+(b)\} = \bigvee_{x=ab} \{1 \wedge \mu_B^+(b)\} \\ &= \bigvee_{x=ab} \mu_B^+(b) \leq \bigvee_{x=ab} \mu_B^+(ab) = \mu_B^+(x). \end{aligned}$$

And

$$\begin{aligned} (\mathcal{S}_\Gamma^- \circ \mu_B^-)(x) &= \bigwedge_{x=ab} \{\mathcal{S}_\Gamma^-(a) \vee \mu_B^-(b)\} = \bigwedge_{x=ab} \{-1 \vee \mu_B^-(b)\} \\ &= \bigwedge_{x=ab} \mu_B^-(b) \geq \bigwedge_{x=ab} \mu_B^-(ab) = \mu_B^-(x). \end{aligned}$$

Thus $\mathcal{S}_\Gamma^+ \circ \mu_B^+ \subseteq \mu_B^+$ and $\mathcal{S}_\Gamma^- \circ \mu_B^- \supseteq \mu_B^-$.

Conversely, let $\mathcal{S}_\Gamma^+ \circ \mu_B^+ \subseteq \mu_B^+$, $\mathcal{S}_\Gamma^- \circ \mu_B^- \supseteq \mu_B^-$ and $x, y \in S$, then

$$\begin{aligned} \mu_B^+(xy) &\geq (\mathcal{S}_\Gamma^+ \circ \mu_B^+)(xy) = \bigvee_{xy=ab} \{\mathcal{S}_\Gamma^+(a) \wedge \mu_B^+(b)\} \\ &\geq \mathcal{S}_\Gamma^+(x) \wedge \mu_B^+(y) = 1 \wedge \mu_B^+(y) = \mu_B^+(y). \end{aligned}$$

And

$$\begin{aligned} \mu_B^-(xy) &\leq (\mathcal{S}_\Gamma^- \circ \mu_B^-)(xy) = \bigwedge_{xy=ab} \{\mathcal{S}_\Gamma^-(a) \vee \mu_B^-(b)\} \\ &\leq \mathcal{S}_\Gamma^-(x) \vee \mu_B^-(y) = -1 \vee \mu_B^-(y) = \mu_B^-(y). \end{aligned}$$

Thus $\mu_B^+(xy) \geq \mu_B^+(y)$ and $\mu_B^-(xy) \leq \mu_B^-(y)$. Thus $B = \langle \mu_B^+, \mu_B^- \rangle$ is a BVF-left ideal of S . The second case can be seen in a similar way. \square

Let $B_1 = \langle \mu_{B_1}^+, \mu_{B_1}^- \rangle$ and $B_2 = \langle \mu_{B_2}^+, \mu_{B_2}^- \rangle$ be two BVF-subsets of an LA-semigroup S . The symbol $B_1 \cap B_2$ will mean the following

$$(\mu_{B_1}^+ \cap \mu_{B_2}^+)(x) = \mu_{B_1}^+(x) \wedge \mu_{B_2}^+(x), \text{ for all } x \in S.$$

$$(\mu_{B_1}^- \cup \mu_{B_2}^-)(x) = \mu_{B_1}^-(x) \vee \mu_{B_2}^-(x), \text{ for all } x \in S.$$

The symbol $A \cup B$ will mean the following

$$(\mu_{B_1}^+ \cup \mu_{B_2}^+)(x) = \mu_{B_1}^+(x) \vee \mu_{B_2}^+(x), \text{ for all } x \in S.$$

$$(\mu_{B_1}^- \cap \mu_{B_2}^-)(x) = \mu_{B_1}^-(x) \wedge \mu_{B_2}^-(x), \text{ for all } x \in S.$$

Theorem 3.3 *Let S be an LA-semigroup and $B_1 = \langle \mu_{B_1}^+, \mu_{B_1}^- \rangle$ be a BVF-right ideal of S and $B_2 = \langle \mu_{B_2}^+, \mu_{B_2}^- \rangle$ be a BVF-left ideal of S , then $B_1 \circ B_2 \subseteq B_1 \cap B_2$.*

Proof. Let for any $x, y, z \in S$, if $x \neq yz$, then we have

$$(\mu_{B_1}^+ \circ \mu_{B_2}^+)(x) = 0 \leq \mu_{B_1}^+(x) \wedge \mu_{B_2}^+(x) = (\mu_{B_1}^+ \cap \mu_{B_2}^+)(x).$$

And

$$(\mu_{B_1}^- \circ \mu_{B_2}^-)(x) = 0 \geq \mu_{B_1}^-(x) \vee \mu_{B_2}^-(x) = (\mu_{B_1}^- \cup \mu_{B_2}^-)(x).$$

Otherwise

$$\begin{aligned} (\mu_{B_1}^+ \circ \mu_{B_2}^+)(x) &= \bigvee_{x=yz} \{ \mu_{B_1}^+(y) \wedge \mu_{B_2}^+(z) \} \\ &\leq \bigvee_{x=yz} \{ \mu_{B_1}^+(yz) \wedge \mu_{B_2}^+(yz) \} \\ &= \{ \mu_{B_1}^+(x) \wedge \mu_{B_2}^+(x) \} \\ &= (\mu_{B_1}^+ \cap \mu_{B_2}^+)(x). \end{aligned}$$

And

$$\begin{aligned} (\mu_{B_1}^- \circ \mu_{B_2}^-)(x) &= \bigvee_{x=yz} \{ \mu_{B_1}^-(y) \vee \mu_{B_2}^-(z) \} \\ &\geq \bigvee_{x=yz} \{ \mu_{B_1}^-(yz) \vee \mu_{B_2}^-(yz) \} \\ &= \{ \mu_{B_1}^-(x) \vee \mu_{B_2}^-(x) \} \\ &= (\mu_{B_1}^- \cup \mu_{B_2}^-)(x). \end{aligned}$$

Thus we get $\mu_{B_1}^+ \circ \mu_{B_2}^+ \subseteq \mu_{B_1}^+ \cap \mu_{B_2}^+$ and $\mu_{B_1}^- \circ \mu_{B_2}^- \supseteq \mu_{B_1}^- \cup \mu_{B_2}^-$. Hence $B_1 \circ B_2 \subseteq B_1 \cap B_2$. \square

Proposition 3.4 Let $B_1 = \langle \mu_{B_1}^+, \mu_{B_1}^- \rangle$ and $B_2 = \langle \mu_{B_2}^+, \mu_{B_2}^- \rangle$ be two BVF-LA-subsemigroups of S . Then $B_1 \cap B_2$ is also a BVF-LA-subsemigroup of S .

Proof. Let $B_1 = \langle \mu_{B_1}^+, \mu_{B_1}^- \rangle$ and $B_2 = \langle \mu_{B_2}^+, \mu_{B_2}^- \rangle$ be two BVF-LA-subsemigroups of S . Let $x, y \in S$. Then

$$\begin{aligned} (\mu_{B_1}^+ \cap \mu_{B_2}^+)(xy) &= \mu_{B_1}^+(xy) \wedge \mu_{B_2}^+(xy) \\ &\geq (\mu_{B_1}^+(x) \wedge \mu_{B_1}^+(y)) \wedge (\mu_{B_2}^+(x) \wedge \mu_{B_2}^+(y)) \\ &= (\mu_{B_1}^+(x) \wedge \mu_{B_2}^+(x)) \wedge (\mu_{B_1}^+(y) \wedge \mu_{B_2}^+(y)) \\ &= (\mu_{B_1}^+ \cap \mu_{B_2}^+)(x) \wedge (\mu_{B_1}^+ \cap \mu_{B_2}^+)(y). \end{aligned}$$

And

$$\begin{aligned} (\mu_{B_1}^- \cup \mu_{B_2}^-)(xy) &= \mu_{B_1}^-(xy) \vee \mu_{B_2}^-(xy) \\ &\leq (\mu_{B_1}^-(x) \vee \mu_{B_1}^-(y)) \vee (\mu_{B_2}^-(x) \vee \mu_{B_2}^-(y)) \\ &= (\mu_{B_1}^-(x) \vee \mu_{B_2}^-(x)) \vee (\mu_{B_1}^-(y) \vee \mu_{B_2}^-(y)) \\ &= (\mu_{B_1}^- \cup \mu_{B_2}^-)(x) \vee (\mu_{B_1}^- \cup \mu_{B_2}^-)(y). \end{aligned}$$

Thus $B_1 \cap B_2$ is also a bipolar-valued fuzzy LA-subsemigroup of S . \square

Proposition 3.5 Let $B_1 = \langle \mu_{B_1}^+, \mu_{B_1}^- \rangle$ and $B_2 = \langle \mu_{B_2}^+, \mu_{B_2}^- \rangle$ be two BVF-left (resp. BVF-right, BVF-two-sided) ideal of S . Then $B_1 \cap B_2$ is also a BVF-left (resp. BVF-right, BVF-two-sided) ideal of S .

Proof. The proof is similar to the proof of Proposition 3.4. \square

Lemma 3.6 In an LA-semigroup S with left identity, for every BVF-left ideal $B = \langle \mu_B^+, \mu_B^- \rangle$ of S , we have $\Gamma \circ B = B$. Where $\Gamma = \langle \mathcal{S}_\Gamma^+(x), \mathcal{S}_\Gamma^-(x) \rangle$.

Proof. Let $B = \langle \mu_B^+, \mu_B^- \rangle$ be a BVF-left ideal of S . It is sufficient to show that $\mathcal{S}_\Gamma^+ \circ \mu_B^+ \subseteq \mu_B^+$ and $\mathcal{S}_\Gamma^- \circ \mu_B^- \supseteq \mu_B^-$. Now $x = ex$, for all x in S , as e is left identity in S . So

$$(\mathcal{S}_\Gamma^+ \circ \mu_B^+)(x) = \bigvee_{x=yz} \{ \mathcal{S}_\Gamma^+(y) \wedge \mu_B^+(z) \} \geq \mathcal{S}_\Gamma^+(e) \wedge \mu_B^+(x) = 1 \wedge \mu_B^+(x) = \mu_B^+(x).$$

And

$$(\mathcal{S}_\Gamma^- \circ \mu_B^-)(x) = \bigwedge_{x=yz} \{ \mathcal{S}_\Gamma^-(y) \vee \mu_B^-(z) \} \leq \mathcal{S}_\Gamma^-(e) \vee \mu_B^-(x) = -1 \vee \mu_B^-(x) = \mu_B^-(x).$$

Thus $\mathcal{S}_\Gamma^+ \circ \mu_B^+ \supseteq \mu_B^+$ and $\mathcal{S}_\Gamma^- \circ \mu_B^- \subseteq \mu_B^-$. Hence $\Gamma \circ B = B$. \square

Definition 3.4 Let S be an LA-semigroup and let $\emptyset \neq A \subseteq S$. Then bipolar-valued fuzzy characteristic function $\chi_A = \langle \mu_{\chi_A}^+, \mu_{\chi_A}^- \rangle$ of A is defined as

$$\mu_{\chi_A}^+ = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad \text{and} \quad \mu_{\chi_A}^- = \begin{cases} -1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Theorem 3.7 Let A be a nonempty subset of an LA-semigroup S . Then A is an LA-subsemigroup of S if and only if χ_A is a BVF-LA-subsemigroup of S .

Proof. Let A be an LA-subsemigroup of S . For any $x, y \in S$, we have the following cases:

Case (1) : If $x, y \in A$, then $xy \in A$. Since A is an LA-subsemigroup of S . Then $\mu_{\chi_A}^+(xy) = 1$, $\mu_{\chi_A}^+(x) = 1$ and $\mu_{\chi_A}^+(y) = 1$. Therefore

$$\mu_{\chi_A}^+(xy) = \mu_{\chi_A}^+(x) \wedge \mu_{\chi_A}^+(y).$$

And $\mu_{\chi_A}^-(xy) = -1$, $\mu_{\chi_A}^-(x) = -1$ and $\mu_{\chi_A}^-(y) = -1$. Therefore

$$\mu_{\chi_A}^-(xy) = \mu_{\chi_A}^-(x) \vee \mu_{\chi_A}^-(y).$$

Case (2) : If $x, y \notin A$, then $\mu_{\chi_A}^+(x) = 0$ and $\mu_{\chi_A}^+(y) = 0$. So

$$\mu_{\chi_A}^+(xy) \geq 0 = \mu_{\chi_A}^+(x) \wedge \mu_{\chi_A}^+(y).$$

And $\mu_{\chi_A}^-(x) = 0$ and $\mu_{\chi_A}^-(y) = 0$. So

$$\mu_{\chi_A}^-(xy) \leq 0 = \mu_{\chi_A}^-(x) \vee \mu_{\chi_A}^-(y).$$

Case (3) : If $x \in A$ or $y \in A$. If $x \in A$ and $y \notin A$, then $\mu_{\chi_A}^+(x) = 1$ and $\mu_{\chi_A}^+(y) = 0$. So

$$\mu_{\chi_A}^+(xy) \geq 0 = \mu_{\chi_A}^+(x) \wedge \mu_{\chi_A}^+(y).$$

Now if $x \notin A$ and $y \in A$, then $\mu_{\chi_A}^+(x) = 0$ and $\mu_{\chi_A}^+(y) = 1$. So

$$\mu_{\chi_A}^+(xy) \geq 0 = \mu_{\chi_A}^+(x) \wedge \mu_{\chi_A}^+(y).$$

And if $x \in A$ or $y \in A$. If $x \in A$ and $y \notin A$, then $\mu_{\chi_A}^-(x) = -1$ and $\mu_{\chi_A}^-(y) = 0$. So

$$\mu_{\chi_A}^-(xy) \leq 0 = \mu_{\chi_A}^-(x) \vee \mu_{\chi_A}^-(y).$$

Now if $x \notin A$ and $y \in A$, then $\mu_{\chi_A}^-(x) = 0$ and $\mu_{\chi_A}^-(y) = -1$. So

$$\mu_{\chi_A}^-(xy) \leq 0 = \mu_{\chi_A}^-(x) \vee \mu_{\chi_A}^-(y).$$

Hence $\chi_A = \langle \mu_{\chi_A}^+, \mu_{\chi_A}^- \rangle$ is a BVF-LA-subsemigroup of S .

Conversely, suppose $\chi_A = \langle \mu_{\chi_A}^+, \mu_{\chi_A}^- \rangle$ is a BVF-LA-subsemigroup of S and let $x, y \in A$. Then we have

$$\begin{aligned} \mu_{\chi_A}^+(xy) &\geq \mu_{\chi_A}^+(x) \wedge \mu_{\chi_A}^+(y) = 1 \wedge 1 = 1 \\ \mu_{\chi_A}^+(xy) &\geq 1 \text{ but } \mu_{\chi_A}^+(xy) \leq 1 \\ \mu_{\chi_A}^+(xy) &= 1 \end{aligned}$$

And

$$\begin{aligned} \mu_{\chi_A}^-(xy) &\leq \mu_{\chi_A}^-(x) \vee \mu_{\chi_A}^-(y) = -1 \vee -1 = -1 \\ \mu_{\chi_A}^-(xy) &\leq -1 \text{ but } \mu_{\chi_A}^-(xy) \geq -1 \\ \mu_{\chi_A}^-(xy) &= -1 \end{aligned}$$

Hence $xy \in A$. Therefore A is an LA-subsemigroup of S . \square

Theorem 3.8 Let A be a nonempty subset of an LA-semigroup S . Then A is a left (resp. right) ideal of S if and only if χ_A is a BVF-left (resp. BVF-right) ideal of S .

Proof. The proof of this theorem is similar to Theorem 3.7. \square

Definition 3.5 A BVF-subset $B = \langle \mu_B^+, \mu_B^- \rangle$ of an LA-semigroup S is called a BVF-generalized bi-ideal of S if

$$\mu_B^+((xy)z) \geq \mu_B^+(x) \wedge \mu_B^+(y) \quad \text{and} \quad \mu_B^-((xy)z) \leq \mu_B^-(x) \vee \mu_B^-(y), \quad \text{for all } x, y, z \in S.$$

Definition 3.6 A BVF-LA-subsemigroup $B = \langle \mu_B^+, \mu_B^- \rangle$ of an LA-semigroup S is called a BVF-bi-ideal of S if

$$\mu_B^+((xy)z) \geq \mu_B^+(x) \wedge \mu_B^+(y) \quad \text{and} \quad \mu_B^-((xy)z) \leq \mu_B^-(x) \vee \mu_B^-(y), \quad \text{for all } x, y, z \in S.$$

Lemma 3.9 A BVF-subset $B = \langle \mu_B^+, \mu_B^- \rangle$ of an LA-semigroup S is a BVF-generalized bi-ideal of S if and only if $(\mu_B^+ \circ \mathcal{S}_\Gamma^+) \circ \mu_B^+ \subseteq \mu_B^+$ and $(\mu_B^- \circ \mathcal{S}_\Gamma^-) \circ \mu_B^- \supseteq \mu_B^-$.

Proof. Let $B = \langle \mu_B^+, \mu_B^- \rangle$ be a BVF-generalized bi-ideal of an LA-semigroup S and $x \in S$. If $((\mu_B^+ \circ \mathcal{S}_\Gamma^+) \circ \mu_B^+)(x) = 0$ and $((\mu_B^- \circ \mathcal{S}_\Gamma^-) \circ \mu_B^-)(x) = 0$, then

$$((\mu_B^+ \circ \mathcal{S}_\Gamma^+) \circ \mu_B^+)(x) = 0 \leq \mu_B^+(x) \quad \text{and} \quad ((\mu_B^- \circ \mathcal{S}_\Gamma^-) \circ \mu_B^-)(x) = 0 \geq \mu_B^-(x).$$

Otherwise

$$\begin{aligned} ((\mu_B^+ \circ \mathcal{S}_\Gamma^+) \circ \mu_B^+)(x) &= \bigvee_{x=ab} \{(\mu_B^+ \circ \mathcal{S}_\Gamma^+)(a) \wedge \mu_B^+(b)\} \\ &= \bigvee_{x=ab} \left\{ \bigvee_{a=mn} \{\mu_B^+(m) \wedge \mathcal{S}_\Gamma^+(n)\} \wedge \mu_B^+(b) \right\} \\ &= \bigvee_{x=ab} \bigvee_{a=mn} \{(\mu_B^+(m) \wedge 1) \wedge \mu_B^+(b)\} \\ &= \bigvee_{x=ab} \bigvee_{a=mn} \{\mu_B^+(m) \wedge \mu_B^+(b)\} \\ &\leq \mu_B^+(x). \end{aligned}$$

And

$$\begin{aligned} ((\mu_B^- \circ \mathcal{S}_\Gamma^-) \circ \mu_B^-)(x) &= \bigwedge_{x=ab} \{(\mu_B^- \circ \mathcal{S}_\Gamma^-)(a) \vee \mu_B^-(b)\} \\ &= \bigwedge_{x=ab} \left\{ \bigwedge_{a=mn} \{\mu_B^-(m) \vee \mathcal{S}_\Gamma^-(n)\} \vee \mu_B^-(b) \right\} \\ &= \bigwedge_{x=ab} \bigwedge_{a=mn} \{(\mu_B^-(m) \vee -1) \vee \mu_B^-(b)\} \\ &= \bigwedge_{x=ab} \bigwedge_{a=mn} \{\mu_B^-(m) \vee \mu_B^-(b)\} \\ &\geq \mu_B^-(x). \end{aligned}$$

Thus $(\mu_B^+ \circ \mathcal{S}_\Gamma^+) \circ \mu_B^+ \subseteq \mu_B^+$ and $(\mu_B^- \circ \mathcal{S}_\Gamma^-) \circ \mu_B^- \supseteq \mu_B^-$.

Conversely, assume that $(\mu_B^+ \circ \mathcal{S}_\Gamma^+) \circ \mu_B^+ \subseteq \mu_B^+$ and $(\mu_B^- \circ \mathcal{S}_\Gamma^-) \circ \mu_B^- \supseteq \mu_B^-$. Let $x, y, z \in S$, then

$$\begin{aligned} \mu_B^+((xy)z) &\geq ((\mu_B^+ \circ \mathcal{S}_\Gamma^+) \circ \mu_B^+)((xy)z) = \bigvee_{(xy)z=cd} \{(\mu_B^+ \circ \mathcal{S}_\Gamma^+)(c) \wedge \mu_B^+(d)\} \\ &\geq (\mu_B^+ \circ \mathcal{S}_\Gamma^+)(xy) \wedge \mu_B^+(z) = \left\{ \bigvee_{xy=pq} \{\mu_B^+(p) \wedge \mathcal{S}_\Gamma^+(q)\} \right\} \wedge \mu_B^+(z) \\ &\geq \{\mu_B^+(x) \wedge \mathcal{S}_\Gamma^+(y)\} \wedge \mu_B^+(z) = \{\mu_B^+(x) \wedge 1\} \wedge \mu_B^+(z) \\ &= \mu_B^+(x) \wedge \mu_B^+(z). \end{aligned}$$

And

$$\begin{aligned}
\mu_B^-((xy)z) &\leq ((\mu_B^- \circ \mathcal{S}_\Gamma^-) \circ \mu_B^-)((xy)z) = \bigwedge_{(xy)z=c d} \{(\mu_B^- \circ \mathcal{S}_\Gamma^-)(c) \vee \mu_B^-(d)\} \\
&\leq (\mu_B^- \circ \mathcal{S}_\Gamma^-)(xy) \vee \mu_B^-(z) = \left\{ \bigwedge_{xy=pq} \{\mu_B^-(p) \vee \mathcal{S}_\Gamma^-(q)\} \right\} \vee \mu_B^-(z) \\
&\leq \{\mu_B^-(x) \vee \mathcal{S}_\Gamma^-(y)\} \vee \mu_B^-(z) = \{\mu_B^-(x) \vee -1\} \vee \mu_B^-(z) \\
&= \mu_B^-(x) \vee \mu_B^-(z).
\end{aligned}$$

Thus $\mu_B^+((xy)z) \geq \mu_B^+(x) \wedge \mu_B^+(z)$ and $\mu_B^-((xy)z) \leq \mu_B^-(x) \vee \mu_B^-(z)$, which implies that $B = \langle \mu_B^+, \mu_B^- \rangle$ is a BVF-generalized bi-ideal of S . \square

Lemma 3.10 *Let $B = \langle \mu_B^+, \mu_B^- \rangle$ be a BVF-subset of an LA-semigroup S then $B = \langle \mu_B^+, \mu_B^- \rangle$ is a BVF-bi-ideal of S if and only if $\mu_B^+ \circ \mu_B^+ \subseteq \mu_B^+$, $\mu_B^- \circ \mu_B^- \supseteq \mu_B^-$, $(\mu_B^+ \circ \mathcal{S}_\Gamma^+) \circ \mu_B^+ \subseteq \mu_B^+$ and $(\mu_B^- \circ \mathcal{S}_\Gamma^-) \circ \mu_B^- \supseteq \mu_B^-$.*

Proof. Follows from Lemma 3.2(1) and Lemma 3.9. \square

Definition 3.7 A BVF-subset $B = \langle \mu_B^+, \mu_B^- \rangle$ of an LA-semigroup S is called a BVF-interior ideal of S if

$$\mu_B^+((xy)z) \geq \mu_B^+(y) \quad \text{and} \quad \mu_B^-((xy)z) \leq \mu_B^-(y), \quad \text{for all } x, y, z \in S.$$

Lemma 3.11 *Let $B = \langle \mu_B^+, \mu_B^- \rangle$ be a BVF-subset of an LA-semigroup S then $B = \langle \mu_B^+, \mu_B^- \rangle$ is a BVF-interior ideal of S if and only if $(\mathcal{S}_\Gamma^+ \circ \mu_B^+) \circ \mathcal{S}_\Gamma^+ \subseteq \mu_B^+$ and $(\mathcal{S}_\Gamma^- \circ \mu_B^-) \circ \mathcal{S}_\Gamma^- \supseteq \mu_B^-$.*

Proof. The proof of this lemma is similar to the proof of Lemma 3.9. \square

Remark 3.12 *Every BVF-ideal is a BVF-interior ideal of an LA-semigroup S , but the converse is not true.*

Example 3.2 Let $S = \{a, b, c, d\}$, the binary operation "·" on S be defined as follows:

| · | a | b | c | d |
|---|---|---|---|---|
| a | c | c | c | d |
| b | d | d | c | c |
| c | d | d | d | d |
| d | d | d | d | d |

Clearly, S is an LA-semigroup. But S is not a semigroup because $c = a \cdot (a \cdot b) \neq (a \cdot a) \cdot b = d$. Now we define BVF-subset as

$$B = \langle \mu_B^+, \mu_B^- \rangle = \left\langle \left(\frac{a}{0.5}, \frac{b}{0.3}, \frac{c}{0.1}, \frac{d}{0.8} \right), \left(\frac{a}{-0.7}, \frac{b}{-0.4}, \frac{c}{-0.2}, \frac{d}{-0.9} \right) \right\rangle.$$

It can be verified that $B = \langle \mu_B^+, \mu_B^- \rangle$ is a BVF-interior ideal of S . But, since

$$\mu_B^+(b \cdot c) = \mu_B^+(c) = 0.1 < 0.3 = \mu_B^+(b).$$

And

$$\mu_B^-(b \cdot c) = \mu_B^-(c) = -0.2 > -0.4 = \mu_B^-(b).$$

Thus $B = \langle \mu_B^+, \mu_B^- \rangle$ is not a BVF-right ideal of S , that is, $B = \langle \mu_B^+, \mu_B^- \rangle$ is not a BVF-two-sided ideal of S .

Proposition 3.13 *Every BVF-subset $B = \langle \mu_B^+, \mu_B^- \rangle$ of an LA-semigroup S with left identity is a BVF-right ideal if and only if it is a BVF-interior ideal.*

Proof. Let every BVF-subset $B = \langle \mu_B^+, \mu_B^- \rangle$ of S is a BVF-right ideal. For x, a and y of S , consider

$$\mu_B^+((xa)y) \geq \mu_B^+(xa) = \mu_B^+((ex)a) = \mu_B^+((ax)e) \geq \mu_B^+(ax) \geq \mu_B^+(a).$$

And

$$\mu_B^-((xa)y) \leq \mu_B^-(xa) = \mu_B^-((ex)a) = \mu_B^-((ax)e) \leq \mu_B^-(ax) \leq \mu_B^-(a).$$

Which implies that $B = \langle \mu_B^+, \mu_B^- \rangle$ is a BVF-interior ideal. Conversely, for any x and y in S we have,

$$\mu_B^+(xy) = \mu_B^+((ex)y) \geq \mu_B^+(x) \quad \text{and} \quad \mu_B^-(xy) = \mu_B^-((ex)y) \leq \mu_B^-(x).$$

Hence required. \square

Theorem 3.14 *Let $B = \langle \mu_B^+, \mu_B^- \rangle$ be a BVF-left ideal of an LA-semigroup S with left identity, then $B = \langle \mu_B^+, \mu_B^- \rangle$ being BVF-interior ideal is a BVF-bi-ideal of S .*

Proof. Since $B = \langle \mu_B^+, \mu_B^- \rangle$ is an BVF-left ideal in S , so $\mu_B^+(xy) \geq \mu_B^+(y)$ and $\mu_B^-(xy) \leq \mu_B^-(y)$ for all x and y in S . As e is left identity in S . So,

$$\mu_B^+(xy) = \mu_B^+((ex)y) \geq \mu_B^+(x) \quad \text{and} \quad \mu_B^-(xy) = \mu_B^-((ex)y) \leq \mu_B^-(x),$$

which implies that $\mu_B^+(xy) \geq \mu_B^+(x) \wedge \mu_B^+(y)$ and $\mu_B^-(xy) \leq \mu_B^-(x) \vee \mu_B^-(y)$ for all x and y in S . Thus $B = \langle \mu_B^+, \mu_B^- \rangle$ is an BVF-LA-subsemigroup of S . For any x, y and z in S , we get

$$\mu_B^+((xy)z) = \mu_B^+((x(ey))z) = \mu_B^+((e(xy))z) \geq \mu_B^+(xy) = \mu_B^+((ex)y) \geq \mu_B^+(x).$$

And

$$\mu_B^-((xy)z) = \mu_B^-((x(ey))z) = \mu_B^-((e(xy))z) \leq \mu_B^-(xy) = \mu_B^-((ex)y) \leq \mu_B^-(x).$$

Also

$$\mu_B^+((xy)z) = \mu_B^+((zy)x) = \mu_B^+((z(ey))x) = \mu_B^+((e(zy))x) \geq \mu_B^+(zy) = \mu_B^+((ez)y) \geq \mu_B^+(z).$$

And

$$\mu_B^-((xy)z) = \mu_B^-((zy)x) = \mu_B^-((z(ey))x) = \mu_B^-((e(zy))x) \leq \mu_B^-(zy) = \mu_B^-((ez)y) \leq \mu_B^-(z).$$

Hence $\mu_B^+((xy)z) \geq \mu_B^+(x) \wedge \mu_B^+(z)$ and $\mu_B^-((xy)z) \leq \mu_B^-(x) \vee \mu_B^-(z)$ for all x, y and z in S . \square

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